

Cauchy Problem for for some high order generalization of Korteweg - de Vries equation

Z A Sobirov, S Abdinazarov

Mechanics and Mathematics Faculty of National University of Uzbekistan

e-mail: sobirovzar@yahoo.com

(Dated: January 26, 2013)

In this work we study Cauchy problem for a high-order differential equation $\frac{\partial u(y,x)}{\partial y} + P\left(\frac{\partial}{\partial x}\right)u(y,x) = \gamma \frac{\partial}{\partial x}(u^2(y,x)) + F(y,x)$. We prove that the problem is well-posed both for linear ($\gamma = 0$) and nonlinear equations on the class of rapidly decaying Schwartz functions. Furthermore, for the case when the initial condition is given on $L_2(\mathbf{R}^1)$ we prove the existence of the unique solution on the space $L_\infty(0, y_0; L_2(\mathbf{R}^1)) \cap L_2(0, y_0; H^{n-1}(\mathbf{R}^1)) \cap L_2(0, y_0; H^n(-r, r))$, where r is an arbitrary positive number. It is also shown that the solution continuously depends on the initial conditions.

Keywords: Nonlinear partial differential equations, weak solution, generalized KdV equation, Green function, decreasing solutions, existence theorems, continuous dependence on initial function

I. INTRODUCTION

Korteweg - de Vries (KdV) equation is a nonlinear differential equation that has important application in different areas of physics(e.g., acoustics, hydrodynamics, optics etc). Therefore studying its properties is of fundamental and practical importance. Especially, the problem of soliton transport in non-uniform media causes special importance of KdV equation. Different practical applications of this equation and the properties of its solutions can be found in the Refs. [1], [2], (see also [7] and references therein).

In this work we treat Cauchy problem for a generalization of Korteweg - de Vries equation. Namely, we consider third order derivative in the KdV with odd order differential operator with constant coefficient.

This paper is organized as follows In the next section we prove solvability of the Cauchy problem in the Schwartz class of rapidly decreasing functions. In particular, subsections A and B deal with the deal with the linear counterpart, for which using Fourier transformation, we show that under certain (necessary and sufficient) condition for the coefficients, the problem has a unique solution in the Schwartz class. The subsection B presents also some estimates for the Green function.

In subsection I.C we treat the nonlinear equation by obtaining countably many set of a-priory estimates implying convergence of the iteration procedure with respect to nonlinear term in some interval $(0; y_1)$; $y_1 > 0$. We also prove that this solution can be continued to an arbitrary interval $(0; y_0)$; $y_0 > y_1$. The case of KdV equation ($n = 1$) one can find in [6].

In the section III we consider Cauchy problem with the initial function in $L_2(\mathbf{R}^1)$ We explore the case of odd and high than third order equation. In the subsection IIIA using Green function method we prove the existence of a weak solution for linear equation. Also, we obtain some a-priory estimates that will be used in the further analysis and show continuity of the obtained solution. The next subsections present the proofs for the existence of weak solution for non linear equation and few a-priory estimates. Finally, using the Green function we prove linear dependence of the weak solution on initial data.

II. SOLVABILITY OF CAUCHY PROBLEM IN THE SCHWARTZ CLASS OF RAPIDLY DECREASING FUNCTIONS.

A. Well-posedness of linear equation.

We consider Cauchy problem for the equation

$$Lu \equiv (-1)^n \frac{\partial u}{\partial y} + \frac{\partial^{2n+1} u}{\partial x^{2n+1}} + \sum_{k=0}^{2n-1} b_k \frac{\partial^k u}{\partial x^k} = f(y, x), \quad (1)$$

$$u(y, x)|_{y=0} = u_0(y), \quad (2)$$

in the half string $D_{y_0} = \{(y, x) : -\infty < x < \infty, 0 < y < y_0\}$ where b_k are constants and $y_0 = \text{const} > 0$.

First we consider the case $f(y, x) \in C^1([0, y_0]; S(\mathbf{R}^1))$, $u_0 \in S(\mathbf{R}^1)$.

Theorem 1. *Let $\sum_{k=0}^{n-1} (-1)^{n+k} b_{2k} \lambda^{2k} \geq 0$ for large enough values of $\lambda > 0$. Then there is unique solution of the Cauchy problem (1), (2) in $C^1([0, y_0]; S(\mathbf{R}^1))$.*

Proof. Using Fourier transform we have

$$\frac{d}{dy} \tilde{u}(y, \lambda) = P(-i\lambda) \tilde{u}(y, \lambda) + \tilde{f}(y, \lambda), \quad \tilde{u}(0, \lambda) = \tilde{u}_0(\lambda), \quad (3)$$

where \tilde{u} , \tilde{f} and \tilde{u}_0 are Fourier image of the functions u , f and u_0 , respectively,

$$P(\lambda) = (-1)^{n+1} \left[\lambda^{2n+1} + \sum_{k=0}^{2n-1} b_k \lambda^k \right].$$

The unique solution of the problem (3) is

$$\begin{aligned} \tilde{u}(y, \lambda) &= \tilde{u}_0(\lambda) \exp(P(-i\lambda)y) + \\ &+ \int_0^y \tilde{f}(\eta, \lambda) \cdot \exp(P(-i\lambda)(y - \eta)) d\eta. \end{aligned} \quad (4)$$

According to the conditions of theorem $\exp(P(-i\lambda)y)$ is bounded for $y > 0$ and its derivatives with respect to λ have at most polynomial growth at $|\lambda| \rightarrow +\infty$. Consequently $\tilde{u}(y, \lambda) \in S(\mathbf{R}^1)$ for any $y \geq 0$. According to properties of direct and Fourier transforms we conclude that Cauchy problem (1), (2) has unique solution $u(y, \lambda) \in C^1([0, y_0]; S(\mathbf{R}^1))$.

B. Fundamental solution. Green function for Cauchy problem

It is known that the solution of model equation

$$(-1)^n \frac{\partial u(y, x)}{\partial y} + \frac{\partial^{2n+1} u(y, x)}{\partial x^{2n+1}} = g(y, x) \quad (5)$$

which satisfies initial condition $u(0, x) = u_0(x)$ given by [3]

$$\begin{aligned} u(y, x) &= \int_{-\infty}^{\infty} U(y, x - \xi) u_0(\xi) d\xi + \\ &+ (-1)^n \int_0^y d\eta \int_{-\infty}^{\infty} U(y - \eta, x - \xi) g(\eta, \xi) d\xi, \end{aligned} \quad (6)$$

where

$$U(y, x) = \pi^{-1} y^{-1/(2n+1)} \text{Ain} \left(xy^{-1/(2n+1)} \right), \quad (7)$$

is a fundamental solution and

$$\text{Ain}(x) = \int_0^\infty \cos(\lambda^{2n+1} - \lambda x) d\lambda \quad (8)$$

is a Airy function which satisfies the following ordinary differential equation

$$\left(\frac{d^{2n}}{dx^{2n}} + \frac{(-1)^n x}{2n+1} \right) z(x) = 0. \quad (9)$$

The fundamental solution satisfies the following estimates

$$\left| \frac{\partial^j U(y, x)}{\partial x^j} \right| \leq c_1 \frac{x^{-\frac{2(n-j)-1}{4n}}}{y^{\frac{2j+1}{4n}}}, \quad j = \overline{0, 2n-1}, \quad (10)$$

for $x > 0$ and

$$\left| \frac{\partial^j U(y, x)}{\partial x^j} \right| \leq \frac{c_2}{y^{j+1} 2n} \exp \left(-c \frac{(-x)^{\frac{2n+1}{2n}}}{y^{\frac{1}{2n}}} \right), \quad j = \overline{0, 2n-1}, \quad (11)$$

for $x < 0$, $c, c_1, c_2 = \text{const} > 0$.

$$\begin{aligned} u(y, x) = & \int_{-\infty}^\infty U(y, x - \xi) u_0(\xi) d\xi + \\ & + (-1)^n \int_0^y d\eta \int_{-\infty}^\infty U(y - \eta, x - \xi) f(\eta \xi) d\xi + \mathcal{J}u(y, x), \end{aligned} \quad (12)$$

where

$$\mathcal{J}u(y, x) = (-1)^n \int_0^y d\eta \int_{-\infty}^\infty \sum_{k=0}^{2n-1} \frac{\partial^k U}{\partial x^k}(y - \eta, x - \xi) u(\eta, \xi) d\xi. \quad (13)$$

According to the theorem 1 integral equation (12) has unique solution. Therefore the solution of Cauchy problem can be presented by the following form

$$\begin{aligned} u(y, x) = & \int_{-\infty}^\infty G(y, x - \xi) u_0(\xi) d\xi + \\ & + (-1)^n \int_0^y d\eta \int_{-\infty}^\infty G(y - \eta, x - \xi) f(\eta \xi) d\xi, \end{aligned} \quad (14)$$

where

$$\begin{aligned} G(y - \eta, x - \xi) = & \int_0^y \int_{-\infty}^\infty R(y - t, x - z) U(t - \eta, z - \xi) dz dt + \\ & + U(y - \eta, x - \xi), \end{aligned}$$

$R(y - \eta, x - \xi)$ is a resolvent of integral operator \mathcal{J} . It is known that resolvent of integral equation satisfies the same estimates at infinity as kernel of integral operator. So,

$$|R(y, x)| \leq \begin{cases} c_1 \frac{x^{\frac{2n-1}{4n}}}{y^{\frac{1}{2n+1}}}, & \frac{x}{y^{1/(2n+1)}} \rightarrow +\infty; \\ \frac{c_2}{y^{\frac{2n}{2n+1}}} \exp \left(-c \frac{(-x)^{\frac{2n+1}{2n}}}{y^{\frac{1}{2n}}} \right), & \frac{x}{y^{1/(2n+1)}} \rightarrow -\infty \end{cases} \quad (15)$$

Proposition. *The function G satisfies the following estimates*

$$\left| \frac{\partial^k}{\partial x^k} G(y, x) \right| \leq \begin{cases} c \cdot \frac{x^{1+\frac{k}{2n}}}{y^{\frac{2k+1}{4n}}} & \text{for } x > 0; \\ \frac{\text{const}}{y^{\frac{k+1}{2n+1}}} \exp\left(-c' \frac{(-x)^{\frac{2n+1}{2n}}}{y^{\frac{1}{2n}}}\right) & \text{for } x < 0, \end{cases} \quad (16)$$

where $k = 0; 1$.

Proof. We put

$$I(y, x) = \int_0^y dt \int_{-\infty}^{\infty} R(y-t, x-z) U(z, t) dz. \quad (17)$$

(a) Let $x > 0$. Then rewrite the integral $\frac{\partial^k}{\partial x^k} I(y, x)$ in the form

$$\begin{aligned} \frac{\partial^k}{\partial x^k} I(y, x) &= \int_0^y dt \int_{-\infty}^{\infty} R(y-t, x-z) \frac{\partial^k}{\partial x^k} U(t, z) dz = \\ &= \int_0^y dt \left[\int_{-\infty}^0 + \int_0^x + \int_x^{+\infty} \right] dz \equiv I_1^{(k)} + I_2^{(k)} + I_3^{(k)}. \end{aligned} \quad (18)$$

Using the estimates for the functions R and U , using substitutions $z = -xz_1$ and $z_1 = y^{\frac{1}{2n+1}} x^{-1} z_2^{\frac{2n}{2n+1}}$ also taking to account the identity

$$\int_0^y (y-t)^{\alpha} t^{\beta} dt = y^{\alpha+\beta-1} B(\alpha, \beta)$$

we have

$$\left| I_1^{(k)} \right| \leq c_3 \frac{x^{\frac{2n-1}{4n}}}{y^{\frac{k+1}{2n+1} - \frac{1}{4n}}} \left[\frac{y^{\frac{1}{2n+1}}}{x} \int_0^{+\infty} z^{-\frac{1}{2n+1}} e^{-cz} dz + \frac{y^{\frac{1}{2n+1}} (1 + \frac{2n-1}{4n})}{x^{1+\frac{2n-1}{4n}}} \int_0^{+\infty} z^{\frac{1}{2}} e^{-cz} dz \right] \leq c_5 \frac{x^{\frac{2n-1}{4n}}}{y^{\frac{2k+1}{4n}}}. \quad (19)$$

Using straightforward calculation we obtain

$$\left| I_2^{(k)} \right| \leq c_6 \int_0^y \int_0^x \frac{(x-z)^{\frac{2n-1}{4n}}}{(y-t)^{\frac{4n-1}{4n}}} \cdot \frac{z^{-\frac{2(n-k)-1}{4n}}}{t^{2k+1} 4n} dz dt = c_7 \frac{x^{1+\frac{k}{2n}}}{y^{\frac{k}{2n}}}. \quad (20)$$

The integral $I_3^{(k)}$ can be estimated analogously as $I_2^{(k)}$

$$\left| I_3^{(k)} \right| \leq c_{11} \frac{x^{-\frac{2(n-k)-1}{4n}}}{y^{\frac{2k+1}{4n} - \frac{2}{2n+1}}} \left[1 + \frac{x^{\frac{2(n-k)-1}{4n}}}{y^{\frac{2(n-k)-1}{4n(2n+1)}}} \right]. \quad (21)$$

From the estimates (19) – (21) we obtain estimation for $\frac{\partial^k}{\partial x^k} G(y, x)$ at $x > 0$. (b) Now we consider the case $x < 0$. We rewrite the integral $\frac{\partial^k}{\partial x^k} I(y, x)$ in the form

$$\begin{aligned} \frac{\partial^k}{\partial x^k} I(y, x) &= \int_0^y dt \int_{-\infty}^{\infty} R(y-t, x-z) \frac{\partial^k}{\partial x^k} U(t, z) dz = \\ &= \int_0^y dt \left[\int_{-\infty}^x + \int_x^0 + \int_0^{+\infty} \right] dz \equiv I_4^{(k)} + I_5^{(k)} + I_6^{(k)}. \end{aligned} \quad (22)$$

Taking to account estimates for the functions U and R and using $\theta^b \exp(-a\theta) \leq \text{const}$ for $\theta > 0, a > 0$ we have

$$\begin{aligned}
|I_4^{(k)}| &\leq \frac{c_{13}}{y^{\frac{k+1}{2n+1}-\frac{1}{4n}-\frac{2n-1}{4n(2n+1)}}} \int_{-\infty}^x \exp\left(- (c-\varepsilon) \frac{(-z)^{\frac{2n+1}{2n}}}{y^{\frac{1}{2n}}}\right) dz = \\
&\quad \left\{ z = z_1^{\frac{2n}{2n+1}} y^{\frac{1}{2n+1}} \right\} = \\
&\quad \frac{2n}{2n+1} \frac{c_{13}}{y^{\frac{k}{2n+1}-\frac{1}{4n}-\frac{2n-1}{4n(2n+1)}}} \int_{-\infty}^{\frac{(-x)^{(2n+1)/(2n)}}{y^{1/(2n)}}} (-z_1)^{-\frac{1}{2n+1}} \exp(c_0 z_1) dz_1 \\
&\leq \frac{c_{14}}{y^{\frac{k}{2n+1}-\frac{1}{4n}-\frac{2n-1}{4n(2n+1)}}} \int_{-\infty}^{\frac{(-x)^{(2n+1)/(2n)}}{y^{1/(2n)}}} \exp((c_0 - \varepsilon) z_1) dz_1 \\
&\leq \frac{c_{14}}{y^{\frac{k}{2n+1}-\frac{1}{4n}-\frac{2n-1}{4n(2n+1)}}} \exp\left(\frac{(-x)^{(2n+1)/(2n)}}{y^{1/(2n)}}\right). \tag{23}
\end{aligned}$$

Now we estimate $|I_5^{(k)}|$.

$$\begin{aligned}
|I_k^{(5)}| &\leq c_{15} \int_0^y \frac{dt}{(y-t)^{\frac{2n}{2n+1}} t^{\frac{k+1}{2n+1}}} \int_x^0 \exp\left(-c \frac{(z-x)^{\frac{2n+1}{2n}}}{(y-t)^{\frac{1}{2n}}} - c \frac{(-z)^{\frac{2n+1}{2n}}}{t^{\frac{1}{2n}}}\right) \\
&\quad \leq c_{16} \frac{-x}{y^{\frac{k}{2n+1}}} \cdot \exp\left(-c' \frac{(-x)^{\frac{2n+1}{2n}}}{y^{\frac{1}{2n}}}\right). \tag{24}
\end{aligned}$$

Here we use inequalities $y-t < y$, $t < y$ and $a^\alpha + b^\alpha \geq \left(\frac{a+b}{2}\right)^\alpha$ for $a > 0, b > 0, \alpha > 0$.

Analogously as in the case of integral $I_4^{(k)}$ we get

$$|I_6^{(k)}| \leq \frac{c_{19}}{y^{\frac{2k+1}{4n}-\frac{2}{2n+1}-\frac{2(n-k)-1}{4n(2n+1)}}} \exp\left(-c_0 \frac{(-x)^{\frac{2n+1}{2n}}}{y^{\frac{1}{2n}}}\right). \tag{25}$$

From the estimates (23) – (25) we get estimates for $\frac{\partial^k}{\partial x^k} G(y, x)$ at $x < 0$.

C. Cauchy problem for nonlinear equation in the class of rapidly decaying functions

In this section we investigate Cauchy problem for nonlinear equation

$$Lu(y, x) = \gamma \frac{\partial}{\partial x} (u^2(y, x)) + F(y, x), \quad x \in \mathbf{R}^1, y > 0 \tag{26}$$

where $F(y, x) \in C^1([0, y_0]; S(\mathbf{R}^1))$, with initial condition $u(0, x) = u_0(x) \in S(\mathbf{R}^1)$.

Theorem 2. (*Uniqueness of solution*) Let $(-1)^{n+k} b_{2k} \geq 0$ ($k = \overline{0, n-1}$). Then the Cauchy problem (26), (2) has at most one solution in $C^1([0, y_0], S(\mathbf{R}^1))$ for any $y_0 > 0$.

Proof. Let suppose that $u_1(y, x)$ and $u_2(y, x)$ are two different solutions of the Cauchy problem for equation (26). Then the function $u(y, x) = u_1(y, x) - u_2(y, x)$ satisfies the equality

$$Lu = 2\gamma u \frac{\partial u_1}{\partial x} - 2\gamma \frac{\partial u_2}{\partial x} \tag{27}$$

and the initial condition $u(0, x) = 0$.

Multiplying both sides of equality (27) by $(-1)^n u(y, x)$ and integrating with respect to variable x from $-\infty$ to $+\infty$

we get

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial y} \int_{-\infty}^{\infty} u^2 dx + \sum_{k=0}^{n-1} b_{2k} \int_{-\infty}^{\infty} \left(\frac{\partial^k u}{\partial x^k} \right) dx = \\
& = 2\gamma \int_{-\infty}^{\infty} u^2 \left[2 \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} \right] dx \leq \\
& \leq \left(2 \sup_x \left| \frac{\partial u_1}{\partial x} \right| + \sup_x \left| \frac{\partial u_2}{\partial x} \right| \right) \int_{-\infty}^{\infty} u^2 dx \leq A_6 \int_{-\infty}^{\infty} u^2 dx.
\end{aligned} \tag{28}$$

According to conditions of theorem we have

$$\frac{\partial}{\partial y} \int_{-\infty}^{\infty} u^2(y, x) dx \leq A_6 \int_{-\infty}^{\infty} u^2(y, x) dx.$$

It follows

$$e^{-A_6 y} \int_{-\infty}^{\infty} u^2(y, x) dx \leq \int_{-\infty}^{\infty} u^2(0, x) dx = 0.$$

or $u(y, x) \equiv 0$. This proves the theorem.

It is known that convergence in $C^1([0, y_0], S(\mathbf{R}^1))$ are defined by countable set of semi-norms (see [5], [6])

$$\|u\|_{k,s,j}^2 = \sup_y \left[\int_{-\infty}^{\infty} \left| \frac{\partial^{k+j} u(y, x)}{\partial x^k \partial y^j} \right|^2 dx + \int_{-\infty}^{\infty} (1+x^2)^s \left| \frac{\partial^j u(y, x)}{\partial y^j} \right|^2 dx \right],$$

where k, s are nonnegative integers and $j = 0; 1$.

For the further results we need the following

Lemma 1 [6]. Suppose $u \in S(\mathbf{R}^1)$ and for some N the inequality $\int_{-\infty}^{\infty} \left(\frac{\partial^N u}{\partial x^N} \right) dx \leq C = \text{const.}$ Then the following inequality is true

$$\begin{aligned}
\int_{-\infty}^{\infty} x^{2m} \left(\frac{\partial^k u}{\partial x^k} \right)^2 dx & \leq C_1(k, m) \left(\int_{-\infty}^{\infty} x^{2m+2} u^2 dx \right)^{\frac{m}{m+1}} + \\
& + C_2(k, m) \left(\int_{-\infty}^{\infty} x^{2m+2} u^2 dx \right)^{\frac{1}{2k+1}}
\end{aligned} \tag{29}$$

for $2mk \leq N$, where $C_1(m, k), C_2(m, k)$ and C are some positive constants, m, k and N are natural numbers.

It is easy to see that from (29) it follows that

$$\int_{-\infty}^{\infty} x^{2m} \left(\frac{\partial^k u}{\partial x^k} \right)^2 dx \leq M_\varepsilon(m, k) + \varepsilon \int_{-\infty}^{\infty} x^{2m+2} u^2 dx, \tag{30}$$

where ε and $M_\varepsilon(m, k)$ are positive constants.

Theorem 3. Let $(-1)^{n+k} b_{2k} > 0$ ($k = \overline{0, n-1}$). Then there exist positive y_1 which depends on the coefficients of the equation (26) and on quantities

$$\int_{-\infty}^{\infty} \left[u_0^2(x) + \left(\frac{d^2 u_0(x)}{dx^2} \right)^2 \right] dx, \quad \sup_y \int_{-\infty}^{\infty} \left[F^2(y, x) + \left(\frac{d^2 F(y, x)}{dx^2} \right)^2 \right] dx$$

such that Cauchy problem (26), (2) has a solution in $C^1([0, y_1]; S(\mathbf{R}^1))$.

Proof. The case $n = 1$ are investigated in [5] and [6]. So, we will consider the case $n > 1$. We construct sequence $\{u_m(y, x) : m \in \mathbf{N}\}$:

$u_1(y, x) = u_0(x)$, $u_m(y, x)$ ($m \geq 2$) is a solution of the following problem

$$Lu_m(y, x) = \gamma \frac{\partial}{\partial x} (u_{m-1}^2(y, x)) + F(y, x), \quad u_m(0, x) = u_0(x). \quad (31)$$

We will show convergence of this sequence in $C^1([0, y_1], S(\mathbf{R}^1))$.

Multiplying both sides of equation (31) by $(-1)^n 2u_m(y, x)$ and integrating in \mathbf{R}^1 we have

$$\begin{aligned} & 2 \sum_{k=0}^{n-1} (-1)^{n+k} b_{2k} \int_{-\infty}^{\infty} \left(\frac{\partial^k u_m(y, x)}{\partial x^k} \right)^2 dx + \frac{\partial}{\partial y} \int_{-\infty}^{\infty} u_m^2(y, x) dx = \\ & = (-1)^n 2\gamma \int_{-\infty}^{\infty} u_m(y, x) \frac{\partial}{\partial x} (u_{m-1}^2(y, x)) dx + (-1)^n 2 \int_{-\infty}^{\infty} u_m(y, x) F(y, x) dx. \end{aligned} \quad (32)$$

First we consider the first integral on the right hand side of the equality

$$\int_{-\infty}^{\infty} u_m(y, x) \frac{\partial}{\partial x} (u_{m-1}^2(y, x)) dx \leq 2 \sup_x |u_{m-1}(y, x)| \cdot \int_{-\infty}^{\infty} |u_m| \left| \frac{\partial u_{m-1}}{\partial x} \right| dx.$$

We should estimate $\sup_x |u_{m-1}(y, x)|$. Taking to account relation $\lim_{x \rightarrow -\infty} u(y, x) = 0$ we have

$$\sup_x |u_{m-1}(y, x)| \leq \left(2 \int_{-\infty}^{\infty} \left| u_{m-1} \frac{\partial u_{m-1}}{\partial x} \right| dx \right)^{\frac{1}{2}} \leq \left(4 \int_{-\infty}^{\infty} |u_{m-1}|^2 dx \int_{-\infty}^{\infty} \left| \frac{\partial u_{m-1}}{\partial x} \right|^2 dx \right)^{\frac{1}{4}} \quad (33)$$

Taking to account (32)-(33) and using Cauchy $\varepsilon ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, Holder inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$ and inequality

$$\int_{-\infty}^{\infty} (v'(x))^2 dx = - \int_{-\infty}^{\infty} v(x) v''(x) dx \leq \left(\int_{-\infty}^{\infty} v^2(x) dx \int_{-\infty}^{\infty} (v''(x))^2 dx \right)^{\frac{1}{2}} \quad (34)$$

we get

$$\begin{aligned} & 2 \sum_{k=0}^{n-1} (-1)^{n+k} b_{2k} \int_{-\infty}^{\infty} \left(\frac{\partial^k u_m(y, x)}{\partial x^k} \right)^2 dx + \frac{\partial}{\partial y} \int_{-\infty}^{\infty} u_m^2(y, x) dx \\ & \leq \varepsilon \int_{-\infty}^{\infty} u_m^2 dx + C_0(\varepsilon) \left[\left(\int_{-\infty}^{\infty} \left(\frac{\partial^2 u_{m-1}}{\partial x^2} \right)^2 dx \right)^2 + \left(\int_{-\infty}^{\infty} u_{m-1}^2 dx \right)^2 \right] + C_1. \end{aligned} \quad (35)$$

Similarly, taking second partial derivative of both sides of equation with respect to x , multiplying both sides of resulting equation by $2(-1)^n \frac{\partial^2 u_m}{\partial x^2}$ then integrating in \mathbf{R}^1 and using similar operation as above we get

$$\begin{aligned} & 2 \sum_{k=0}^{n-1} (-1)^{n+k} b_{2k} \int_{-\infty}^{\infty} \left(\frac{\partial^{k+2} u_m(y, x)}{\partial x^k} \right)^2 dx + \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \left(\frac{\partial^2 u_m(y, x)}{\partial x^2} \right)^2 dx \\ & \leq \varepsilon \int_{-\infty}^{\infty} \left(\frac{\partial^3 u_m}{\partial x^3} \right)^2 dx + \varepsilon \int_{-\infty}^{\infty} \left(\frac{\partial^2 u_m}{\partial x^2} \right)^2 dx + C_2(\varepsilon) \left[\left(\int_{-\infty}^{\infty} \left(\frac{\partial^2 u_{m-1}}{\partial x^2} \right)^2 dx \right)^2 + \left(\int_{-\infty}^{\infty} u_{m-1}^2 dx \right)^2 \right] + C_3 \end{aligned} \quad (36)$$

Finally we have

$$\frac{\partial}{\partial y} \int_{-\infty}^{\infty} \left[\left(\frac{\partial^2 u_m}{\partial x^2} \right)^2 + u_m^2 \right] dx \leq C_4 \left[\int_{-\infty}^{\infty} \left[\left(\frac{\partial^2 u_{m-1}}{\partial x^2} \right)^2 + u_{m-1}^2 \right] dx \right]^2 + C_5, \quad (37)$$

where C_4 is a constant that depends on the coefficients b_2, b_0, γ of equation (26) and

$$C_5 = \frac{1}{\varepsilon} \sup_y \int_{-\infty}^{\infty} \left(F^2(y, x) + \left(\frac{\partial^2 F(y, x)}{\partial x^2} \right)^2 \right) dx$$

Let suppose

$$\sup_y \int_{-\infty}^{\infty} \left[\left(\frac{\partial^2 u_{m-1}}{\partial x^2} \right)^2 + u_{m-1}^2 \right] dx \leq 2 \int_{-\infty}^{\infty} \left[\left(\frac{d^2 u_0}{dx^2} \right)^2 + u_0^2 \right] dx + C_5. \quad (38)$$

Then one can easily check that

$$\sup_y \int_{-\infty}^{\infty} \left[\left(\frac{\partial^2 u_m}{\partial x^2} \right)^2 + u_m^2 \right] dx \leq 2 \int_{-\infty}^{\infty} \left[\left(\frac{d^2 u_0}{dx^2} \right)^2 + u_0^2 \right] dx + C_5. \quad (39)$$

for

$$y \leq y_1 = \left(4C_4 \left(\int_{-\infty}^{\infty} \left[\left(\frac{d^2 u_0}{dx^2} \right)^2 + u_0^2 \right] dx \right)^2 + 2C_5^2 + C_5 \right)^{-1} \cdot \left(\int_{-\infty}^{\infty} \left[\left(\frac{d^2 u_0}{dx^2} \right)^2 + u_0^2 \right] dx + C_5 \right).$$

Since the inequality (39) is true for $m = 1$ and y_1 does not depend on m we conclude that it is hold for any $m \in \mathbf{N}$, $0 \leq y \leq y_1$.

Now we carry out mathematical induction with respect to order of derivative. Let suppose that $\sup_y \int_{-\infty}^{\infty} \left(\frac{\partial^j u_m}{\partial x^j} \right)^2 dx \leq C$ for any $j < l$, $m \in \mathbf{N}$, where C is constant which is does not dependent on m .

Using similar operation as above and taking to account inequality (39) one can easily obtain

$$\frac{\partial}{\partial y} \int_{-\infty}^{\infty} \left(\frac{\partial^l u_m}{\partial x^l} \right)^2 dx \leq M_4 \int_{-\infty}^{\infty} \left(\frac{\partial^l u_{m-1}}{\partial x^l} \right)^2 dx + M_5$$

which implies $\int_{-\infty}^{\infty} \left(\frac{\partial^l u_m}{\partial x^l} \right)^2 dx \leq \text{const}$ for all $0 \leq y \leq y_1$.

Analogously can be shown that $\int_{-\infty}^{\infty} x^{2p} u_m^2 dx \leq \text{const}$. Then according to lemma 1 we have

$$\int_{-\infty}^{\infty} x^{2p} \left(\frac{\partial^k u_m}{\partial x^k} \right)^2 dx \leq \int_{-\infty}^{\infty} x^{2p+2} u_m^2 dx + A_4 \leq \text{const} < \infty.$$

According to the estimates given above the sequence $\{u_m\}$ converges in $C^1([0, y_1]; S(\mathbf{R}^1))$. It can be easily checked that the function $u := \lim_{n \rightarrow +\infty} u_n$ is a solution of Cauchy problem (26), (2).

Now we will show solvability of Cauchy problem in $0 < y < y_0$ for any $y_0 > 0$.

Theorem 4. Suppose $(-1)^{n+k} b_{2k} > 0$ ($k = \overline{0, n-1}$), $u_0(x) \in S(\mathbf{R}^1)$ and $F(y, x) \in C^1([0, y_0]; S(\mathbf{R}^1))$. Then Cauchy problem (26), (2) has a solution in $C^1([0, y_0]; S(\mathbf{R}^1))$.

Proof. We show that the solution obtained above can be continued to the interval $0 < y < y_0$ for any $y_0 > 0$. According the proof of Theorem 3 it is enough to estimate the norm of u in $L_\infty([0, y_0]; H^3(\mathbf{R}^1))$. Multiplying equation (26) by $(-1)^{n+2k}u(x, y)$, integrating in $D_y = (0, y) \times \mathbf{R}^1$ and integrating by part we have

$$2 \sum_{k=0}^{n-1} (-1)^{n+k} b_{2k} \int_{D_y} (\partial_x^k u(\eta, y))^2 dx d\eta + (1 - \varepsilon) \int_{-\infty}^{\infty} u^2(y, x) dx \leq \int_{-\infty}^{\infty} u_0^2(x) dx + C(\varepsilon, y_0) \int_{D_{y_0}} F^2(y, x) dx dy := l_1(y_0, \varepsilon),$$

where $\partial_x := \frac{\partial}{\partial x}$. It follows

$$\int_{D_y} (\partial_x^k u(\eta, x)) dx d\eta \leq \frac{1}{(-1)^{n+k} b_{2k}} l_1(y_0, \varepsilon), \quad \int_{-\infty}^{\infty} u^2(y, x) dx \leq l_1(y_0, \varepsilon)$$

$k = \overline{0, n-1}$, $0 < y < y_0$.

Taking derivative ∂_x from both sides of equation (26)

$$\begin{aligned} & 2 \sum_{k=0}^{n-1} (-1)^{n+k} b_{2k} \int_{D_y} (\partial_x^{k+1} u(\eta, y))^2 dx d\eta + (1 - \varepsilon) \int_{-\infty}^{\infty} (\partial_x u(y, x))^2 dx \\ & \leq 4(-1)^n \gamma \int_{D_y} (\partial_x u(\eta, x))^3 dx d\eta + \int_{-\infty}^{\infty} (u'_0)^2(x) dx + C(\varepsilon, y_0) \int_{D_{y_0}} (\partial_x F(y, x))^2 dx dy \end{aligned} \quad (40)$$

$$\begin{aligned} & \int_{D_y} (\partial u(\eta, x))^3 dx d\eta \leq \int_0^y \sup_x |\partial u(\eta, x)| \int_{-\infty}^{\infty} (\partial u(\eta, x))^2 dx d\eta \leq \\ & \int_0^y \left(\int_{-\infty}^{\infty} (\partial_x u(\eta, x))^2 dx \right)^{\frac{5}{4}} \left(\int_{-\infty}^{\infty} (\partial_x^2(\eta, x))^2 dx \right)^{\frac{1}{4}} \\ & \leq c_6(\varepsilon) \int_0^y \left(\int_{-\infty}^{\infty} (\partial_x(\eta, x))^2 dx \right)^{\frac{5}{3}} d\eta + \varepsilon \int_0^y \int_{-\infty}^{\infty} (\partial_x u(x, \eta))^2 dx d\eta \\ & \leq \varepsilon \sup_{\eta \in (0, y_0)} \int_{-\infty}^{\infty} (\partial_x u(\eta, x))^2 dx + \varepsilon \int_0^y \int_{-\infty}^{\infty} (\partial_x^2 u(\eta, x))^2 dx + c_8(\varepsilon, y_0). \end{aligned} \quad (41)$$

According inequalities (40) and (41) we have

$$\int_{D_y} (\partial_x^{k+1} u(\eta, x)) dx d\eta \leq \text{const}, \quad \int_{-\infty}^{\infty} (\partial u(y, x))^2 dx \leq \text{const}, \quad k = \overline{0, n-1}, \quad 0 < y < y_0. \quad (42)$$

Analogously one can get

$$\sup_{y \in (0, y_0)} \int_{-\infty}^{\infty} (\partial_x^2 u(y, x))^2 dx \leq C.$$

That proves the theorem.

It should be noticed that if $F(y, x) \in C^\infty([0, y_0]; S(\mathbf{R}^1))$ the Cauchy problem has a solution in $C^\infty([0, y_0]; S(\mathbf{R}^1))$.

III. WEAK SOLUTION OF THE CAUCHY PROBLEM

In this section we investigate Cauchy for the equation

$$Lu = \gamma \partial_x(u^2) \quad (43)$$

$(y, x) \in D_{y_0} := (0, y_0) \times \mathbf{R}^1$, $y_0 > 0$, with initial condition

$$u(y, x)|_{y=0} = u_0(x) \in L_2(\mathbf{R}^1). \quad (44)$$

Definition. The function $u(y, x) \in L_2(D_{y_0})$ is called to be weak solution of Cauchy problem (43), (44) if the following conditions are hold

(a) for any function $\varphi(y, x) \in C_0^\infty(D_{y_0})$

$$\int \int_{D_{y_0}} (u \cdot L^* \varphi + \gamma u^2 \partial_x \varphi) dx dy = 0, \quad (45)$$

where

$$L^* = -\partial^{2n+1} - (-1)^n \partial_y + \sum_{k=0}^{n-1} (-1)^k b_k \partial_x^k;$$

(b) There exist a set $E \subset (0, y_0)$, $\text{mes} E = 0$ such that for any $y \in (0, y_0) \setminus E$ the function $u(y, x)$ is well-defined a.e. in \mathbf{R}^1 and for any function $\omega(x) \in C_0^\infty(\mathbf{R}^1)$ the following equality

$$\lim_{\substack{y \rightarrow 0 \\ y \in (0, y_0) \setminus E}} \int_{-\infty}^{\infty} u(y, x) \omega(x) dx = \int_{-\infty}^{\infty} u_0(x) \omega(x) dx \quad (46)$$

is hold.

Further in this section we will define by $C(\cdot, \cdot, \dots)$ different positive constants than depends on entering parameters.

For any function $v(x) \in L_2(\mathbf{R}^1)$ and $\alpha > 0$ we put

$$\|v\| = \int_{-\infty}^{\infty} v^2(x) dx, \quad N_\alpha(v) = \int_{-\infty}^0 |x|^\alpha v^2(x) dx.$$

Let $\psi_0(x) \in C^\infty(\mathbf{R}^1)$ is a non decreasing function such that $\psi_0(x) = 0$ for $x \leq \frac{1}{2}$, $\psi_0(x) = 1$ for $x \geq 0$ and it strictly increase in $\frac{1}{2} \leq x \leq 1$. For $\alpha > 0$ we put $\psi_\alpha := x^\alpha \cdot \psi_0(x)$. It is easy to see that $\psi_\alpha(x) \in C^\infty(\mathbf{R}^1)$ and $\psi'_\alpha \geq 0$.

First we will consider the case of linear equation ($\gamma = 0$).

A. Well-posedness of Cauchy problem for linear equation

In this subsection we will consider the case $\gamma = 0$ i.e. the case of linear equation.

Theorem 5. Let $(-1)^{n+k} b_{2k} \geq 0$ ($k = \overline{0, n-1}$) and there exist constant $\varepsilon > 0$ such that $N_{3+\varepsilon}(u_0(x)) < \infty$. Then

$$u(y, x) = \int_{-\infty}^{\infty} G(y, x - \xi) u_0(\xi) d\xi \quad (47)$$

is a unique weak solution of the Cauchy problem and

$$\text{ess sup}_{0 < y < y_0} \|u(y, x)\| \leq c \|u_0\|. \quad (48)$$

Furthermore, if for some $\alpha > 0$ the quantity $N_\alpha(u_0)$ is bounded and $(-1)^{n+k}b_{2k} > 0$ ($k = \overline{0, n-1}$) then

$$\operatorname{ess\,sup}_{0 < y < y_0} N_\alpha(u(y, x)) \leq C(\alpha, y_0) [\|u_0\| + N_\alpha(u_0)]. \quad (49)$$

Proof. Theorem 1 implies that adjoint Cauchy problem with initial condition in $S(\mathbf{R}^1)$ has unique solution in $C^1([0, y_0]; S(\mathbf{R}^1))$. Then according to [8] – [10] we conclude that Cauchy problem (43), (44) has unique solution in the class of function adjoint to $C^1([0, y_0]; S(\mathbf{R}^1))$ given by (47).

Now it is enough to establish estimates (48), (49). We put

$$u_0^h(x) := \frac{1}{h} \int_{-\infty}^{\infty} \lambda\left(\frac{x-\xi}{h}\right) u_0(\xi) d\xi,$$

where $h > 0$, $\lambda(x) \geq 0$, $\lambda(x) \in C_0^\infty(\mathbf{R}^1)$, $\operatorname{supp} \lambda(x) \subseteq [-1, 1]$, $\int_{-\infty}^{\infty} \lambda(x) dx = 1$.

$$u_{0h}(x) := u_0^h(x) \cdot \psi_0(x + 1/h) \cdot \psi_0(1/h - x), \quad 0 < h < 1.$$

We will investigate the following problem

$$Lw_h(y, x) = 0, \quad w_h|_{y=0} = u_{0h}(x) \in C_0^\infty(\mathbf{R}^1). \quad (50)$$

The function $w_h(y, x) = \int_{-\infty}^{\infty} G(y, x - \xi) u_{0h}(\xi) d\xi$ is a solution of this problem and according to result of previous section $w_h \in C^\infty(0, y_0; S(\mathbf{R}^1))$ and $\|w_h\| \leq \|u_0\|$.

Multiplying both sides of equation (50) by $w_h(y, x)\psi_\alpha(1-x)$ and integrating in \mathbf{R}^1 we have

$$\begin{aligned} & \frac{1}{2} \int_{-\infty}^{\infty} w_h^2(y, x) \psi_\alpha(1-x) dx + \frac{2n+1}{2n} \int_0^y \int_{-\infty}^{\infty} (\partial_x^n w_h)^2 \psi'_\alpha(1-x) dx d\eta + \\ & + \sum_{k=1}^{n-1} (-1)^{n+k} b_{2k} \int_0^y \int_{-\infty}^{\infty} (\partial_x^k w_h)^2 \psi_\alpha(1-x) dx d\eta \leq \int_{-\infty}^{\infty} u_{0h}^2(x) \psi_\alpha(1-x) dx + \\ & + \sum_{k=0}^{n-1} \sum_{1 \leq m+2k \leq 2n+1} C_{km} \int_0^y \int_{-\infty}^{\infty} (\partial_x^k w_h)^2 \psi_\alpha^{(m)}(1-x) dx d\eta \end{aligned} \quad (51)$$

where C_{mk} are constants that depends on m and the coefficients of equation.

First we consider the case $0 < \alpha \leq 1$. Then $|\psi_\alpha^{(k)}(x)| \leq C(\alpha)$ for all $k = \overline{1, 2n+1}$. Therefore, inequality (51) implies

$$\int_0^y \int_{-\infty}^{\infty} (\partial_x^k w_h)^2 \psi_\alpha(1-x) dx d\eta \leq C(\alpha, y_0) \cdot \|u_0\|^2, \quad k = \overline{1, n-1},$$

$$\int_{-\infty}^{\infty} w_h^2(y, x) \psi_\alpha(1-x) dx \leq C(\alpha, y_0) \cdot \|u_0\|^2.$$

Let suppose that the inequalities

$$\int_0^y \int_{-\infty}^{\infty} (\partial_x^k w_h)^2 \psi_\alpha(1-x) dx d\eta \leq C(\alpha, y_0) \cdot [\|u_0\|^2 + N_\alpha(u_0)], \quad k = \overline{1, n-1}, \quad (52)$$

$$\int_{-\infty}^{\infty} w_h^2(y, x) \psi_\alpha(1-x) dx \leq C(\alpha, y_0) \cdot [\|u_0\|^2 + N_\alpha(u_0)]. \quad (53)$$

are hold for $0 < \alpha \leq p$, p is positive integer.

Then for $p < \alpha \leq p+1$ taking to account inequality $|\psi_\alpha^{(k)}| \leq C(\alpha) (1 + \psi_{\alpha-1}(x))$ we get

$$\begin{aligned} & \frac{1}{2} \int_{-\infty}^{\infty} w_h^2(y, x) \psi_\alpha(1-x) dx + \frac{2n+1}{2n} \int_0^y \int_{-\infty}^{\infty} (\partial_x^n w_h)^2 \psi'_\alpha(1-x) dx d\eta + \\ & + \sum_{k=1}^{n-1} (-1)^{n+k} b_{2k} \int_0^y \int_{-\infty}^{\infty} (\partial_x^k w_h)^2 \psi_\alpha(1-x) dx d\eta \leq \\ & \leq C(\alpha, y_0) \sum_{k=0}^{n-1} \sum_{1 \leq m+2k \leq 2n+1} \int_0^y \left[\int_{-\infty}^{\infty} (\partial_x^k w_h)^2 dx + \int_{-\infty}^{\infty} \psi_{\alpha-1}(1-x) (\partial_x^k w_h)^2 dx \right] d\eta + \\ & + \int_{-\infty}^{\infty} u_0^2(x) \psi_\alpha(1-x) dx \leq C(\alpha, y_0) [\|u_0\|^2 + N_\alpha(u_0)]. \end{aligned} \quad (54)$$

From the last inequality one can conclude that the inequalities (52), (53) are true for any positive integer p . The estimates (48) and (49) follow from inequalities $\|w_h\| \leq \|u_0\|$ and (53).

Theorem 6. (Continuity of the solution) *Let $\sum_{k=0}^{n-1} (-1)^{n+k} b_{2k} \lambda^k \geq 0$ for large enough values of λ and there exist constant $\varepsilon > 0$ such that $N_{3+\varepsilon}(u_0(x)) < \infty$. Then the solution $u(y, x)$ defined by (47) is a continuous in any interior point of the domain D_{y_0} . Moreover, for any $x_0 \in \mathbf{R}^1$, $0 < y < y_0$ the following inequality is hold*

$$\text{ess sup}_{x \leq x_0} |u(x, y)| \leq C(\varepsilon, y_0, x_0, \|u_0\|, N_{3+\varepsilon}(u_0)) \cdot y^{-\frac{1}{4n}}. \quad (55)$$

Proof. According to estimates (16) and Cauchy-Bunyakowsky inequality we have

$$\begin{aligned} & \left| \int_{-\infty}^x G(y, x-\xi) u_0(\xi) d\xi \right| \leq C \int_{-\infty}^x \frac{x-\xi+1}{y^{\frac{1}{4n}}} |u_0(\xi)| d\xi \leq \\ & \leq \frac{C(\varepsilon)}{y^{\frac{1}{4n}}} \left(\int_{-\infty}^{x_0} u_0^2(\xi) (x_0 - \xi + 1)^{3+\varepsilon} d\xi \right)^{\frac{1}{2}} \leq \frac{C(\varepsilon, x_0, \|u_0\|, N_\alpha(u_0))}{y^{\frac{1}{4n}}}. \end{aligned} \quad (56)$$

$$\begin{aligned} & \left| \int_x^\infty G(y, x-\xi) u_0(\xi) d\xi \right| \leq \int_x^\infty |u_0(\xi)| \frac{1}{y^{\frac{1}{2n+1}}} \exp \left(-c_0 \frac{(\xi-x)^{\frac{2n+1}{2n}}}{y^{\frac{1}{2n}}} \right) d\xi \leq \\ & \leq \frac{c}{y^{\frac{1}{2n+1}}} \left(\int_x^\infty u_0^2(\xi) d\xi \right)^{\frac{1}{2}} \cdot \left(\int_x^\infty \exp \left[-2c_0 \frac{(\xi-x)^{\frac{2n+1}{2n}}}{y^{\frac{1}{2n}}} \right] d\xi \right)^{\frac{1}{2}} \leq \frac{C\|u_0\|}{y^{\frac{1}{4n+2}}}. \end{aligned} \quad (57)$$

The inequalities (56), (57) implies the estimate (55). It is not difficult to see that the condition of theorem are sufficient for uniform convergence of the integral in (47). Hence $u(y, x)$ is continuous.

B. Existence of weak solution for non linear equation

In this paragraph we investigate existence of weak solution in the sense of definition 1.

First we give estimates for the case when initial condition is in $S(\mathbf{R}^1)$.

Lemma 1. Let $(-1)^{n+k}b_{2k} > 0$, $k = \overline{0, n-1}$. Then the following estimates are hold

$$\sup_{0 < y < y_0} \|u(y, x)\| \leq \|u_0(x)\|, \quad (58)$$

$$\|u\|_{L_2(0, y_0; H^{n-1}(\mathbf{R}^1))} \leq C\|u_0\|, \quad (59)$$

$$\int_0^{y_0} \int_{x_0-1}^{x_0} (\partial_x^n u(y, x))^2 dx dy \leq C(y_0, \|u_0\|), \quad (60)$$

for all $x_0 \in \mathbf{R}^1$. Furthermore, if $N_\alpha(u_0) \leq \infty$ for some $\alpha \geq 0$ then

$$\sup_{0 < y < y_0} N_\alpha(u(y, x)) \leq C(\alpha, y_0, \|u_0\|, N_\alpha(u_0)), \quad (61)$$

$$\int_0^{y_0} N_\alpha(\partial_x^k u(y, x)) dy \leq C(\alpha, y_0, \|u_0\|, N_\alpha(u_0)), \quad k = \overline{1, n-1}, \quad (62)$$

and if $\alpha > 0$ then

$$\int_0^{y_0} (x_0 - x + 1)^{\alpha-1} (\partial_x^n u(y, x))^2 dx dy \leq C(\alpha, y_0, x_0, \|u_0\|, N_\alpha(u_0)). \quad (63)$$

Proof. The estimates (58) and (59) was proven in the previous section. To show other estimates we multiply both sides of equation (43) by $u(y, x) \cdot \psi_\alpha(x_0 - x)$ and integrate in D_y

$$\begin{aligned} & \frac{2n+1}{2} \int_0^y \int_{-\infty}^{\infty} (\partial_x^n u(y, x))^2 \psi'_\alpha(x_0 - x) dx d\eta + \\ & + \sum_{k=0}^{n-1} \int_0^y \int_{-\infty}^{\infty} (\partial_x^k u)^2 \psi_\alpha(x_0 - x) dx d\eta + \frac{1}{2} \int_{-\infty}^{\infty} u^2 \psi_\alpha(x_0 - x) dx d\eta \\ & \leq \sum_{k=0}^{n-1} \int_0^y \int_{-\infty}^{\infty} (\partial_x^k u)^2 \sum_{j=1}^{2n+1-2k} A_{kj} |\psi_\alpha^{(j)}(x_0 - x)| dx d\eta + \\ & + \frac{1}{2} \int_{-\infty}^{\infty} u_0^2(x) \psi_\alpha(x_0 - x) dx + |\gamma| \int_0^y \int_{-\infty}^{\infty} |u|^3 \psi'_\alpha(x_0 - x) dx d\eta. \end{aligned} \quad (64)$$

Initially we estimate third integral on the right hand side of (64)

$$\int_{-\infty}^{\infty} |u(y, x)|^3 \psi'_\alpha(x_0 - x) dx \leq \sup_x |u(y, x) \sqrt{\psi'_\alpha(x_0 - x)}| \int_{-\infty}^{\infty} u^2(y, x) \sqrt{\psi'_\alpha(x_0 - x)} dx.$$

Using inequality (34) we estimate supreme

$$\sup_x |u(y, x) \sqrt{\psi'_\alpha(x_0 - x)}| \leq \sqrt{2} \left(\int_{-\infty}^{\infty} |u \sqrt{\psi'_\alpha}| \cdot \left| \partial_x u \sqrt{\psi'_\alpha} + \frac{u \cdot \psi''_\alpha}{\sqrt{\psi'_\alpha}} \right| \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{\gamma} \left(\int_{-\infty}^{\infty} u^2 \cdot |\psi''_{\alpha}| dx \right)^{\frac{1}{2}} + \sqrt{2}\gamma \left(\int_{-\infty}^{\infty} u^2 \psi'_{\alpha} dx \right)^{\frac{1}{4}} \left(\int_{-\infty}^{\infty} (\partial_x u)^2 \psi'_{\alpha} dx \right)^{\frac{1}{4}}.$$

Continuing inequality (64) we get

$$\begin{aligned} & \frac{2n+1}{2} \int_0^y \int_{-\infty}^{\infty} (\partial_x^n u(y, x))^2 \psi'_{\alpha}(x_0 - x) dx d\eta + \\ & + \sum_{k=0}^{n-1} \int_0^y \int_{-\infty}^{\infty} (\partial_x^k u)^2 \psi_{\alpha}(x_0 - x) dx d\eta + \frac{1}{2} \int_{-\infty}^{\infty} u^2 \psi_{\alpha}(x_0 - x) dx d\eta \\ & \leq \sum \int_0^y \int_{-\infty}^{\infty} (\partial_x^k u)^2 \sum_{j=1}^{2n+1-2k} A_{kj} |\psi_{\alpha}^{(j)}(x_0 - x)| dx d\eta + \frac{1}{2} \int_{-\infty}^{\infty} u_0^2(x) \psi_{\alpha}(x_0 - x) dx + \\ & + \int_0^y \left(\int_{-\infty}^{\infty} u^2 \cdot |\psi''_{\alpha}| dx \right)^{\frac{1}{2}} dy + \sqrt{2}\gamma^2 \int_0^y \left(\int_{-\infty}^{\infty} u^2 \psi'_{\alpha} dx \right)^{\frac{1}{4}} \left(\int_{-\infty}^{\infty} (\partial_x u)^2 \psi'_{\alpha} dx \right)^{\frac{1}{4}} dy \end{aligned} \quad (65)$$

Now we will use mathematical induction method. Let $0 \leq \alpha \leq 1$. Then $|\psi_{\alpha}^{(k)}(x)| \leq C(k)$, $k \geq 1$. Consequently we have

$$\begin{aligned} & \frac{2n-1}{2} \int_0^y \int_{-\infty}^{\infty} (\partial_x^n u(y, x))^2 \psi'_{\alpha}(x_0 - x) dx d\eta + \sum_{k=0}^{n-1} \int_0^y \int_{-\infty}^{\infty} (\partial_x^k u)^2 \psi_{\alpha}(x_0 - x) dx d\eta + \\ & + \frac{1}{2} \int_{-\infty}^{\infty} u^2 \psi_{\alpha}(x_0 - x) dx d\eta \leq C(y_0, \|u_0\|). \end{aligned} \quad (66)$$

From the last inequality one can easily get inequalities (60) – (63).

Suppose that the inequalities (60) – (63) are hold for $0 < \alpha \leq k$ ($k \in \mathbf{N}$).

Let $k < \alpha \leq k+1$. Taking to account the inequalities $|\psi_{\alpha}^{(k)}(x)| \leq C(\alpha)(1 + \psi_{\alpha-1}(x))$, $N_{\alpha-1}(v) \leq N_{\alpha}(v) + \|v\|^2$ we obtain

$$\begin{aligned} & \frac{2n+1}{2} \int_0^y \int_{-\infty}^{\infty} (\partial_x^n u(y, x))^2 \psi'_{\alpha}(x_0 - x) dx d\eta + \sum_{k=0}^{n-1} \int_0^y \int_{-\infty}^{\infty} (\partial_x^k u)^2 \psi_{\alpha}(x_0 - x) dx d\eta + \\ & + \frac{1}{2} \int_{-\infty}^{\infty} u^2 \psi_{\alpha}(x_0 - x) dx d\eta \leq C(\alpha, y_0, x_0, \|u_0\|, N_{\alpha}(u_0)). \end{aligned} \quad (67)$$

which implies inequalities (60) – (63).

Theorem 7. Let $u_0(x) \in L_2(\mathbf{R}^1)$ and $(-1)^{n+k} b_{2k} > 0$, $k = \overline{0, n-1}$. Then there exist a weak solution $u(y, x)$ of Cauchy problem in the sense of definition 1 which is in

$$L_{\infty}(0, y_0; L_2(\mathbf{R}^1)) \cap L_2(0, y_0; H^{n-1}(\mathbf{R}^1)) \cap L_2(0, y_0; H^n(-r, r))$$

for any $r > 0$. The solution $u(y, x)$ satisfies estimates (58) – (63) in lemma 1. Moreover

$$\lim_{y \rightarrow 0} \|u(y, x) - u_0(x)\| = 0. \quad (68)$$

Proof. The case $n = 1$ is considered in [7]. The proof of the theorem for $n > 1$ is almost same as in [7].

Let $\lambda(x) \in C_0^\infty(\mathbf{R}^1)$, $\lambda(x) \geq 0$, $\text{supp}\lambda(x) \subset [-1, 1]$, $\int_{-\infty}^{\infty} \lambda(x)dx = 1$.

We put $u_{0h}(x) := \psi_0(x + 1/h)\psi_0(1/h - x)u_0^h(x)$ where

$$u_0^h(x) = \frac{1}{h} \int_{-\infty}^{\infty} \lambda\left(\frac{x-z}{h}\right) u(z)dz.$$

It is easy to see that $u_{0h}(x) \in C_0^\infty(\mathbf{R}^1)$.

We consider the Cauchy problem

$$Lu_h(y, x) = \partial_x(u_h^2(y, x)), \quad u_h(0, x) = u_{0h}(x). \quad (69)$$

Put $I_r = (-r, r)$ and $Q_r = (0, y_0) \times I_r$. According to inequality $\|u_{0h}(x)\| \leq \|u_0(x)\|$ we have

$$\sup_y \|u_h(y, x)\| \leq \|u_0(x)\|, \quad (70)$$

$$\int_{Q_r} (\partial_x^n u_h(y, x))^2 dx dy \leq C(y_0, r, \|u_0(x)\|), \quad (71)$$

$$\int_0^y \int_{-\infty}^{\infty} (\partial_x^k u_h(y, x))^2 dx dy \leq C\|u_0\|^2, \quad k = \overline{1, n-1}. \quad (72)$$

$$\int_0^{y_0} \left(\sup_{x \in I_r} |u_h(y, x)| \right)^4 dy \leq C(y_0, \|u_0\|) \left[1 + \int_{Q_r} (\partial_x u_h(y, x))^2 dx dy \right] \leq C(y_0, r, \|u_0\|). \quad (73)$$

For $n > 1$ the constant in the last inequality does not depends on r .

Using the last estimates and Holder's inequality we have

$$\begin{aligned} & \int_0^{y_0} \left(\int_{-r}^r u_h^2(y, x) (\partial_x u_h(y, x)) dx \right)^{\frac{2}{3}} dy \leq \\ & \left(\int_0^{y_0} \sup_{x \in I_r} |u_h(y, x)|^4 dy \right)^{\frac{1}{3}} \cdot \left(\int_0^{y_0} \int_{-r}^r (\partial_x u_h(y, x))^2 dx dy \right)^{\frac{2}{3}} \leq C(y_0, r, \|u_0\|). \end{aligned} \quad (74)$$

The inequality (74) means that $u_h \cdot \partial_x u_h$ is bounded in $L_{\frac{4}{3}}(0, y_0; L_2(I_r))$. Consequently we conclude that the set of functions $\{u_h(y, x), h > 0\}$ is bounded in

$$W_r := \{v(y, x) : v \in L_2(0, y_0; H^n(I_r)), \partial_y v \in L_{\frac{4}{3}}(0, y_0; H^{-n-1}(I_r))\}.$$

According to the theorem 5.1 in the first chapter of [4] the set of functions W_r is compact in $L_2(0, y_0; H^{n-1}(I_r))$. So we can select the sequence h_m , $\lim_{m \rightarrow \infty} h_m = 0$ such that $u_{h_m} \rightarrow u$ *-weakly in $L_\infty(0, y_0; L_2(\mathbf{R}^1))$, weakly in $L_2(0, y_0; H^{n-1}(\mathbf{R}^1))$ and for any $r > 0 : u_{h_m} \rightarrow u$ in $L_2(0, y_0; H^{n-1}(-r, r))$, weakly in $L_2(0, y_0; H^n(-r, r))$.

Using the convergence properties given above one can easily show that the function $u(y, x)$ satisfies the integral identity (45) and the estimates (58)–(63).

Now we should consider fulfillment of initial condition. For any positive integer p we define the set $E_p \subset (0, y_0)$, $\text{mes}E_p = 0$ such that for $y \in (0, y_0) \setminus E_p$ the following relations are hold

$$\|u(y, x)\| \leq \|u_0(x)\|, \quad \int_{-p}^p |u_{h_m}(y, x) - u(y, x)|^2 dx \rightarrow 0.$$

We put $E = \bigcup_{p=1}^{\infty} E_p$, $\text{mes}E_p = 0$.

Let $\omega(x) \in C_0^{\infty}(\mathbf{R}^1)$. There exist positive integer p that $\text{supp}\omega(x) \in (-p, p)$. We have

$$\begin{aligned} \left| \int_{-\infty}^{\infty} (u_{h_m}(y, x) - u_0(x))\omega(x)dx \right| &= \left| \int_0^y \int_{-\infty}^{\infty} \partial_y(u_{h_m}(\eta, x))\omega(x)dx d\eta \right| \leq \\ &\leq \|\omega(x)\|_{H^{n+1}(-p, p)} \cdot \int_0^y \|\partial_y u_{h_m}(\eta, x)\|_{H^{-n-1}(-p, p)} d\eta \leq \\ &\leq \|\omega(x)\|_{H^{n+1}(-p, p)} \cdot \left[\|\partial_y u_{h_m}(y, x)\|_{L_{4/3}(0, y_0; H^{-n-1}(-p, p))} \right]^{\frac{3}{4}} \cdot y^{\frac{1}{4}} \leq \\ &\leq C(y_0, p, \|u_0\|) \|\omega(x)\|_{H^{n+1}(-p, p)} \cdot y^{\frac{1}{4}}. \end{aligned}$$

Consequently taking limits $m \rightarrow \infty$, $y \rightarrow 0$ we get (46).

Now we should show (68).

$$\begin{aligned} \|u(y, x) - u - 0(x)\|^2 &\leq 2\|u_0\|^2 - 2 \int_{-\infty}^{\infty} u(y, x) \cdot u_0(x)dx = \\ &+ 2 \int_{-\infty}^{\infty} (u_0(x) - u(y, x))u_{0h}(x)dx + 2 \int_{-\infty}^{\infty} (u_0(x) - u(y, x))(u_0(x) - u_{0h}(x))dx \leq \\ &+ 2 \int_{-\infty}^{\infty} (u_0(x) - u(y, x))u_{0h}(x)dx + 4\|u_0(x)\| \cdot \|u_0(x) - u_{0h}(x)\|. \end{aligned}$$

The last inequality proves (68).

The theorem 7 is proved.

C. Continuous Dependence of The Weak Solution on Initial Data.

For any $\alpha > 0$ any $r > 0$ we put

$$\rho_{\alpha}(x) = \begin{cases} e^{-x}, & x > 0, \\ (1-x)^{\alpha}, & x \leq 0, \end{cases}$$

$$\rho_{\alpha,r}(x) = \begin{cases} \rho_\alpha(x), & x \geq -r, \\ (1+r)^\alpha, & x < -r. \end{cases}$$

$$\gamma_{\alpha,r}(y, \xi, \eta) = \int_{-\infty}^{\infty} (\partial_x(y - \eta, x - \xi))^2 \rho_{\alpha,r}(x) dx.$$

First we give the following lemmas without proof.

Lemma 2. *Let $u(y, x)$ be a weak solution of the Cauchy problem (43), (44) and for some $\varepsilon > 0$*

$$\operatorname{ess\,sup}_y [\|u(y, x)\|^2 + N_{3+\varepsilon}(u(y, x))] \leq \infty.$$

Then the following identity is hold a.e. in D_{y_0}

$$u(y, x) = \int_{-\infty}^{\infty} G(y, x - \xi) u_0(\xi) d\xi + (-1)^n \gamma \int_0^y \int_{-\infty}^{\infty} u^2(y - \eta, x - \xi) d\xi d\eta. \quad (75)$$

Lemma 3. *Let $0 \leq \eta \leq y \leq y_0$. Then the following estimates are hold*

$$\gamma_{\alpha,r}(y, \xi, \eta) \leq C(\alpha, y_0) \rho_{\alpha,r}(\xi) (y - \eta)^{-\frac{4}{2n+1}} \quad \text{for } \xi > 0, \quad (76)$$

$$\gamma_{\alpha,r}(y, \xi, \eta) \leq C(\alpha, y_0) \rho_{\alpha,r}(\xi) (y - \eta)^{-\frac{4}{2n+1}} (-\xi)^{3+\frac{1}{n}} \quad \text{for } \xi < 0. \quad (77)$$

The proof of lemmas 2 and 3 are analogous as similar lemmas in [7].

We define the class of functions

$$\mathbf{K} := \{u(y, x) : M[u, y_0] \leq \infty\},$$

where

$$M[u, y_0] := \operatorname{ess\,sup}_y [\|u(y, x)\|^2 + N_{3+\frac{1}{n}}(u(y, x))].$$

Theorem 8. Let the functions $u(y, x) \in \mathbf{K}$ and $v(y, x) \in \mathbf{K}$ be weak solutions of the Cauchy problem with initial conditions $u(0, x) = u_0(x)$ and $v(0, x) = v_0(x)$ and for some $\alpha_0 \geq 0$ the quantities $N_{\alpha_0}(u_0(x))$ and $N_{\alpha_0}(v_0(x))$ are bounded. Then for any $0 < \alpha \leq \alpha_0$

$$\operatorname{ess\,sup}_y \int_{-\infty}^{\infty} \rho_\alpha(x) (u(y, x) - v(y, x))^2 dx \leq C(\alpha, y_0, M[u, y_0], M[v, y_0]) [\|u_0(x) - v_0(x)\|^2 + N_\alpha(u_0(x) - v_0(x))], \quad (78)$$

and if $\alpha = 0$ then

$$\operatorname{ess\,sup}_y \int_{-\infty}^{\infty} \rho_\alpha(x) (u(y, x) - v(y, x))^2 dx \leq C(\alpha, y_0, M[u, y_0], M[v, y_0]) \cdot \|u_0(x) - v_0(x)\|^2. \quad (79)$$

Proof. Let $\omega(y, x)$ be a solution of Cauchy problem for the equation $L\omega(y, x) = 0$ with initial condition $\omega(0, x) = u_0(x) - v_0(x)$ and $W(y, x) = u(y, x) - v(y, x)$. According to lemma 2 we have

$$W(y, x) = \omega(y, x) + (-1)^n \gamma \int_0^y \int_{-\infty}^{\infty} [u(\eta, \xi) + v(\eta, \xi)] W(\eta, \xi) \partial_x G(y - \eta, x - \xi) d\xi d\eta. \quad (80)$$

Using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ from (80) we have

$$\begin{aligned} & \int_{-\infty}^{\infty} W^2(y, x) \rho_{\alpha, r}(x) dx \leq 2 \int_{-\infty}^{\infty} \omega^2(y, x) \rho_{\alpha, r}(x) dx + \\ & + 2\gamma^2 \int_{-\infty}^{\infty} \rho_{\alpha, r}(x) dx \left(\int_0^y \int_{-\infty}^{\infty} (\partial_x G(y - \eta, x - \xi)) W(\eta, \xi) [u(\eta, \xi) + v(\eta, \xi)] d\xi d\eta \right)^2 dx =: 2I_1 + 2\gamma^2 I_2. \end{aligned} \quad (81)$$

According to the results from previous section we get

$$I_1 = \int_{-\infty}^{\infty} \omega^2(y, x) \rho_{\alpha, r}(x) dx \leq C(\alpha) [\|\omega\|^2 + N_\alpha(\omega)] \leq C(\alpha, y_0) [\|u_0 - v_0\|^2 + N_\alpha(u_0 - v_0)]. \quad (82)$$

To estimate the integral I_2 we will use Cauchy-Bunyakowski inequality.

$$I_2 \leq C(\delta, y_0) \int_{-\infty}^{\infty} \rho_{\alpha, r}(x) \int_0^y (y - \eta)^\delta \nu(\eta) \int_{-\infty}^{\infty} (\partial_x G(y - \eta, x - \xi)) \rho_{\alpha, r}^{-1}(\xi) [u(\eta, \xi) + v(\eta, \xi)]^2 d\xi d\eta dx,$$

where $\frac{1}{3} < \delta < 1$, $\nu(\eta) = \int_{-\infty}^{\infty} W^2(\eta, \xi) \rho_{\alpha, r}(\xi) d\xi$.

Changing integration order and the Lemma 3 we have

$$\begin{aligned} I_2 & \leq C(\delta, y_0) \int_0^y (y - \eta)^\delta \nu(\eta) d\eta \int_{-\infty}^{\infty} [u(\eta, \xi) + v(\eta, \xi)]^2 \rho_{\alpha, r}(\xi) \gamma_{\alpha, r}(y, \xi, \eta) d\xi \\ & \leq 2C(\delta, y_0) \int_0^y (y - \eta)^\delta \nu(\eta) \left[\int_0^{\infty} [u^2(\eta, \xi) + v^2(\eta, \xi)] d\xi + \int_{-\infty}^0 (-\xi)^{3+\frac{1}{n}} [u^2(\eta, \xi) + v^2(\eta, \xi)] d\xi \right] d\eta. \end{aligned} \quad (83)$$

Summarizing inequalities (81) – (83) we obtain

$$\nu(y) \leq C(\alpha, y_0) [\|W\|^2 + N_\alpha(W)] + C(\alpha, y_0, M[u, y_0], M[v, y_0]) \int_0^y (y - \eta)^{\delta - \frac{4}{2n+1}} \nu(\eta) d\eta.$$

Solving the last inequality we have

$$\text{ess sup}_y \int_{-\infty}^{\infty} \rho_{\alpha, r}(x) W(y, x) dx \leq C(\alpha, y_0, M[u, y_0], M[v, y_0]) [\|u_0(x) - v_0(x)\|^2 + N_\alpha(u_0(x) - v_0(x))].$$

Right hand side of the inequality does not depends on r . Therefore we can take limit $r \rightarrow 0$ and obtain the inequalities (78) and (79) for $\alpha = 0$. The theorem 8 is proved.

Thus in this paper we studied Cauchy problem for a high order generalization of KdV equation. In particular, we proved solvability of the problem for the case of initial function in $S(R1)$. In addition, using this result the existence

of a weak solution in the case of initial function in $L_2(R^1)$ and its continuous dependence on the initial conditions are shown.

We used Green function, Fourier transform, iteration, averaging of function, a-priori estimates are used to obtain the above results. Finally, it follows from the theorems 7 and 8 that if $M[u_0; y] = M_0 < +\infty$, then the Cauchy problem has a solution in the class K and in this class the solution is unique.

-
- [1] Gardner C.S. et al.(1967) Method for solving KdV equation. Phys. Rev. Letters, v.19, 1095.
 - [2] Miura R.M., Gardner C.S., Kruskal M.D. (1968) KdV equation and generalizations: II Existence of conservation laws and constants of motions. J. Math. Phys., v.9, 1204.
 - [3] T.D.Djuraev, S.Abdinazarov. (1991) Dokladi of Acad. of Sci. USSR, v.320, 6, p. 1305-1309.
 - [4] Lions J.-L., Moskva, Mir, 1972.
 - [5] Shabat, A.B.(1973) On KdV equation. Doklady Acad. Sci. USSR, 211, 6, 1310.
 - [6] Yakupov V.M. (1975) On Cauchy problem for KdV equation. J. Diff. Equ., 11, 3, 556.
 - [7] Krujkov S.N., Faminskii A.V. (1983) Matematicheskiiy Sbornik, 120(162), 3, 396-425.
 - [8] Gelfand I.M., Shilov G.E. (1953) Uspekhi Math. Nauk USSR,3-54.
 - [9] Gelfand I.M., Shilov G.E. (1955) Docladi Acad. Nauk USSR, 102:6, 1065-1068.
 - [10] Gelfand I.M., Shilov G.E. Generalized functions. Some problems of the theory of differential equations. Moskow, FizMatGIZ, 1958.
 - [11] S.Abdinazarov, Z.A.Sobirov. Cauchy problem for a nonlinear, high odd order equation with multiple characteristics. Proc. of Int. Conf. "Spec. Theory of Diff. Operators and Related Problems ". Sterlitamak, Russia, 2003. p. 71.
 - [12] S.Abdinazarov, Z.A.Sobirov. On continuous dependent of generalized solution of Cauchy problem from initial data for high odd order nonlinear equation. Proc. Int. Russian-Uzbek symposium. Nalchik, 2003, p.10.
 - [13] S.Abdinazarov, Z.A.Sobirov. Cauchy problem for high odd order equation on space. Proc. of Int. Conf. "PDE and related problems of analyses and informatics". Tashkent, 2004 . vol. I. p. 145.